§3.9 Antiderivatives (Anti-D)

Key points: ① Definition of anti-D and the most general anti-D of $f(x)$.

② Finding anti-D using derivative table and linear rules.

③ Velocity and position as anti-D.

Definition: If $F'(x)=f(x)$, then $F(x)$ is ONE ANTIDERIVATIVE of $f(x)$.

$F(x)+C$ is called THE MOST GENERAL ~~[illegible]~~ ANTIDERIVATIVE of $f(x)$, where C is arbitrary constant.

e.g. 1. $(x^2)'=2x$, $2x$ is the derivative of x^2; x^2 is one anti-D of $2x$.

$(x^2+5)'=2x$, $2x$ is the derivative of x^2+5; x^2+5 is one anti-D of $2x$.

For any constant C, $(x^2+C)'=2x$, $2x$ is the derivative of x^2+C, x^2+C is one anti-D of $2x$.

x^2+C is called the most general anti-D of $2x$.

• (Anti) derivative table:

	$F(x)$	$f(x)=F'(x)$		
☆	x^n	$n\cdot x^{n-1}$	$n\cdot x^{n-1}$ has anti-D x^n	$\boxed{x^n \text{ has anti-D } \frac{1}{n+1}x^{n+1}}$ $(n\neq -1)$
	$\sin x$	$\cos x$	$\cos x$ has anti-D $\sin x$	$\cos x$ has anti-D $\sin x$
	$\cos x$	$-\sin x$	$-\sin x$ has anti-D $\cos x$	$\boxed{\sin x \text{ has anti-D } -\cos x}$
	$\tan x$	$\sec^2 x$	$\sec^2 x$ has anti-D $\tan x$	
	$\sec x$	$\tan x\cdot\sec x$	$\tan x\sec x$ has anti-D $\sec x$	

• Linear rule: If $f(x)$ has anti-D $F(x)$, $g(x)$ has anti-D $G(x)$, then

$$a\cdot f(x)+b\cdot g(x) \text{ has anti-D } aF(x)+b\cdot G(x)$$

eg. 2. Find one anti-D of (a) $f(x)=2x^5$, (b) $f(x)=\frac{\sin x}{2}$, (c) $f(x)=2x^5+\frac{\sin x}{2}$.

(a): $f(x)=2\cdot\frac{1}{6}\cdot\boxed{6x^5}$. Notice $(x^6)'=6\cdot x^5 \Rightarrow$ anti-D of $f(x)$ is $F(x)=2\cdot\frac{1}{6}\cdot x^6$

(b): $f(x)=\frac{\sin x}{2}=\frac{(-1)}{2}\cdot(-\sin x)$. $(\cos x)'=-\sin x \Rightarrow$ anti-D of $f(x)$ is $F(x)=\frac{-1}{2}\cdot\cos x$.

(c) According to (a), (b). $2x^5+\frac{\sin x}{2}$ has one anti-D $2\cdot\frac{1}{6}x^6+\frac{-1}{2}\cos x$.

☆ Key anti-D formula: $\boxed{x^n \xrightarrow{\text{anti-D}} \frac{1}{n+1}x^{n+1}}$, $n \neq -1$

eg 3. Find one anti-DF for the following functions f:

(a): $f(x)=1 \Rightarrow F(x)=x$; (a'): $f(x)=-\frac{1}{3} \Rightarrow F(x)=-\frac{1}{3}\cdot x$. (formula with $n=0$)

(b): $f(x)=5x^1 \Rightarrow F(x)=5\cdot\frac{1}{1+1}\cdot x^{1+1}=5\cdot\frac{1}{2}x^2$ (formula with $n=1$)

(c): $f(t)=t^3 \Rightarrow F(t)=\frac{1}{3+1}\cdot t^{3+1}=\frac{1}{4}\cdot t^4$ (formula with $n=3$)

Remark: The formula is also applied to negative n and fraction n.

☆ (d): $f(x)=\frac{1}{x^2} \Rightarrow f(x)=x^{-2} \Rightarrow F(x)=\frac{1}{-2+1}\cdot x^{-2+1}=-x^{-1}=\frac{-1}{x}$ ($n=-2$)

☆ (e): $f(t)=2\sqrt{t} \Rightarrow f(t)=2\cdot t^{\frac{1}{2}} \Rightarrow F(t)=2\cdot\frac{1}{\frac{1}{2}+1}\cdot t^{\frac{1}{2}+1}=2\cdot\frac{1}{\frac{3}{2}}\cdot t^{\frac{3}{2}}=\frac{4}{3}\cdot t^{\frac{3}{2}}$ ($n=\frac{1}{2}$)

☆ According to the definition of Anti-D, the (most general) anti-D of $f'(x)$ is $f(x)+C$. With extra condition on $f(x)$, we can determine the value of C.

eg 4. Suppose $f'(x)=\sin x$ and $f(\frac{\pi}{2})=0$. Find $f(x)$.

Solution: $f(x)$ is the anti-D of $f'(x)=\sin x$. Therefore, $f(x)=-\cos x+C$

Furthermore, plug $x=\frac{\pi}{2}$ into $f(x)=-\cos x+C$, we have,

$f(\frac{\pi}{2})=-\cos\frac{\pi}{2}+C \Leftrightarrow 0=-0+C$ since $f(\frac{\pi}{2})=0$, $\cos\frac{\pi}{2}=0$

$\Rightarrow C=0$ plug into $f(x)=-\cos x+C$.

$\boxed{f(x)=-\cos x}$

Remark: When you get the expression for $f(x)$, it is convinient to double check your answer by computing $f'(x)$ and $f(\frac{\pi}{2})$.

- Moving particle. Position: $S(t)$. Velocity: $V(t)$. Acceleration: $a(t)$

Relation: $S'(t)=V(t)$, $V'(t)=a(t)$

$S(t)$ is the anti-D of $V(t)$; $V(t)$ is the anti-D of $a(t)$

Related problems: Give $V(t)$, find $S(t)$. Give $a(t)$, find $V(t)$.

e.g. 5. A particle is moving along a line with acceleration given by $a(t)=4t^3+2\sin t$.
(f16). Given the initial ~~value~~ velocity is $V(0)=5$ m/s, find the velocity at time $t=\pi$ seconds.

Hint: V is the (general) anti-D of $a(t)$. Find the general anti-D of $4t^3+2\sin t$. Then use the initial condition to determine the constant C

Solution: $t^3 \xrightarrow{\text{anti-D}} \frac{1}{3+1} t^{3+1} = \frac{1}{4} t^4 \ (n=3)$; $\sin t \xrightarrow{\text{anti-D}} -\cos t$.

The general anti-D of $a(t)=4t^3+2\sin t$ is $4\left(\frac{1}{4}t^4\right)+2(-\cos t)+C$

i.e. $\boxed{V(t)} = 4\left(\frac{1}{4}t^4\right)+2(-\cos t)+C = \boxed{t^4-2\cos t+C}$

Plug in $t=0$: $5=V(0)=0^4-2\cos 0+C=0-2+C$ since $\cos 0=1$.

$\Rightarrow 5=-2+C \Rightarrow C=7$. plug back into V's expression.

$\boxed{V(t)=t^4-2\cos t+7}$

Then evaluate V at $t=\pi$, i.e., $\boxed{\begin{aligned} V(\pi)&=\pi^4-2\cdot\cos\pi+7 \\ &=\pi^4+9 \text{ m/s}\end{aligned}}$; $\cos\pi=-1$.

Hints for WW.

※3. Rewrite $f(x)=\dfrac{7-5x^9}{x^3}=\dfrac{7}{x^3}-\dfrac{5x^9}{x^3}=7\cdot x^{-3}-5\cdot x^6$

※4. $y=f(x)$ goes through $(1,0)$ means $f(1)=0$.

The slope of the tangent line $=f'(x)=\dfrac{6}{x^3}-\dfrac{9}{x^5}=6x^{-3}-9\cdot x^{-5}$

Then use the method in e.g. 4 to find $f(x)$.

※5. $\dfrac{1}{\sqrt[3]{x}}=x^{-\frac{1}{3}}$, $\sqrt[3]{x^2}=x^{\frac{2}{3}}$. Apply the anti-D formula with $n=-\frac{1}{3}$ and $n=\frac{2}{3}$

※6. $(x+4)^n$ has anti-D $\dfrac{1}{n+1}(x+4)^{n+1}$. For example, $(x+4)^3 \xrightarrow{\text{anti-D}} \frac{1}{4}(x+4)^4$.

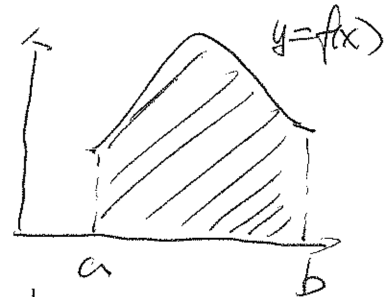
$(x+4)^4 \xrightarrow{\text{anti-D}} \frac{1}{5}(x+4)^5$

※7. $1 \text{ mph}=\dfrac{1 \text{ mile}}{1 \text{ hour}}=\dfrac{22}{15}$ ft/second.

Decelerate at 26 ft/s means $a(t)=-26$ ft/s² $\Rightarrow V(t)=-26\cdot t$ ft/s

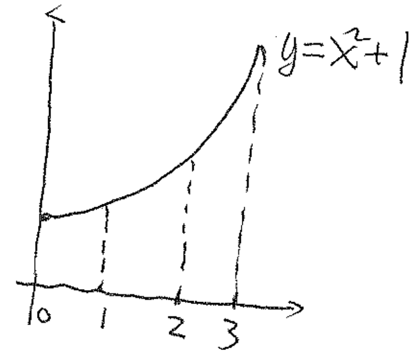
§4.1 Area and distance

Goal: Give $y=f(x)$ from $x=a$ to $x=b$.
We want to find the area under the graph $y=f(x)$ from $x=a$ to $x=b$.



- Estimate by several rectangles of equal width.

eg. $y=x^2+1$ from $x=0$ to $x=3$
Divide $[0,3]$ into three intervals
Set the following table



	x	0	1	2	3
(Height)	$y=x^2+1$	1	2	5	10

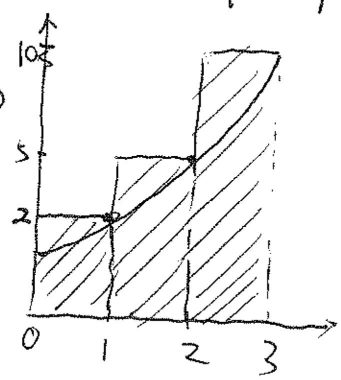
Three intervals $[0,1]$, $[1,2]$, $[2,3]$

left endpoints

right endpoints

Upper sum using right endpoints of each interval.

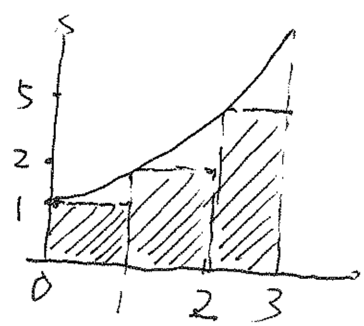
$$1\cdot 2 + 1\cdot 5 + 1\cdot 10 = 1\cdot(2+5+10) = 17.$$



Each rectangle has equal width $\Delta x = \frac{3-0}{3} = 1$.

Lower sum using left endpoints of each interval.

$$1\cdot 1 + 1\cdot 2 + 1\cdot 5 = 1\cdot(1+2+5) = 8$$



Remark:
The actual area ~~of the~~ under the graph should be in between 8 and 17.

If we divide the whole interval more subintervals (thinner and thinner) and repeat the same process, we may get better estimate.

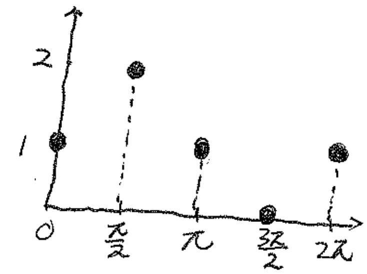
eg 1. Find the upper and lower sum, when we estimate the area
(#15) under the graph of $f(x)=1+\sin x$ from $x=0$ to $x=2\pi$ using four rectangles of equal width.

Width of each interval: $\frac{2\pi-0}{4}=\frac{\pi}{2}$

Solution: whole interval: $[0,2\pi]$. Rectangles: 4.

(Heights) Table:

x	0	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π
$f(x)$	1	2	1	0	1



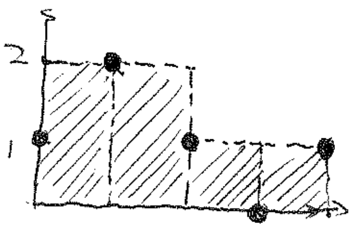
Upper sum

$$=\frac{\pi}{2}\cdot f(\tfrac{\pi}{2})+\frac{\pi}{2}f(\tfrac{\pi}{2})+\frac{\pi}{2}f(\pi)+\frac{\pi}{2}\cdot f(2\pi)=\frac{\pi}{2}(2+2+1+1)=3\pi.$$

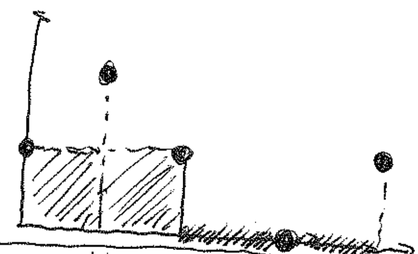
Lower sum

$$=\frac{\pi}{2}\cdot f(0)+\frac{\pi}{2}f(\pi)+\frac{\pi}{2}f(\tfrac{3\pi}{2})+\frac{\pi}{2}f(\tfrac{3\pi}{2})=\frac{\pi}{2}(1+1+0+0)=\pi.$$

Upper sum:



Lower sum:



Remark: Left and Right sum are not necessarily the upper/lower sum.
In eg 1. left sum $=\frac{\pi}{2}[f(0)+f(\frac{1}{2}\pi)+f(\pi)+f(\frac{3\pi}{2})]$, right sum $=\frac{\pi}{2}[f(\frac{\pi}{2})+f(\pi)+f(\frac{3\pi}{2})+f(2\pi)]$.

- Distance is the area ~~of~~ under the graph of $v(t)$. Can be estimated by rectangle method.

eg 2. Give the following table of $v(t)$. Estimate the distance traveled from $t=0$ to $t=10$ using 5 time intervals.

t	0	2	4	6	8	10
$v(t)$	56	55	52	45	31	3

Left endpoints estimate $=2\cdot(56+55+52+45+31)$

Right endpoints estimate $=2\cdot(55+52+45+31+3)$.

Hint for ww 5. Relative velocity = area under the graph of $a(t)$.
Make the table for $a(t)$ and compute the sum of areas for rectangles.

§4.1 Part II. § Appendix E Sigma Notation

end with → n

sum → $\displaystyle\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n$

start with → $i=m$

i is the index.

eg.1 $f(i) = 2i$, $m=1$, $\displaystyle\sum_{i=1}^{n} f(i) = \sum_{i=1}^{n} 2i = 2\cdot1 + 2\cdot2 + 2\cdot3 + \cdots + 2\cdot(n-1) + 2\cdot n$

eg.2 $m=1$, $n=5$, $\displaystyle\sum_{i=1}^{5} \cos(i\pi) = \cos\pi + \cos2\pi + \cos3\pi + \cos4\pi + \cos5\pi$

$= -1 + 1 - 1 + 1 - 1 = -1$

Some formulas in the formula sheet.

eg3 Evaluate the sum $\displaystyle\sum_{k=1}^{43}(7k-8) = \sum_{k=1}^{43} 7k - \sum_{k=1}^{43} 8$

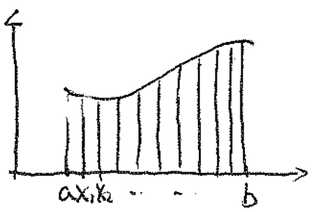
$\displaystyle = 7\cdot\sum_{k=1}^{43} k - \sum_{k=1}^{43} 8$

Formula: $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

$\displaystyle = 7\cdot\frac{43\cdot(43+1)}{2} - 8\cdot43 = 6278$

Sigma Notation is helpful to rewrite the sum for the area problem when we have n rectangles.

eg4



Right endpoints Table: $\Delta x = \dfrac{b-a}{n}$, x_i is the right endpoint of ith interval.

x	x_1	x_2	......	x_n
$f(x)$	$f(x_1)$	$f(x_2)$		$f(x_n)$

$x_i = a + i\cdot\dfrac{b-a}{n}$

Right endpoints sum: $\displaystyle \Delta x\cdot f(x_1) + \Delta x\cdot f(x_2) + \cdots + \Delta x\cdot f(x_n) = \sum_{i=1}^{n} \Delta x\cdot f(x_i)$

As we have more and more thinner rectangles, (i.e., as n tends to infinity), the above sum approaches the actual area under the graph

Use limit notation, we have

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot\Delta x$$, we call such sum the Riemann sum.

eg.5. Calculate the area under the curve $f(x)=2x$ from $x=2$ to $x=3$ by using the Riemann sum $\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot\Delta x$, where $x_i, \Delta x$ are as follow

(a) $[3, 6]$ is divided into n equal subintervals of length Δx

$$\Delta x = \frac{6-3}{n} = \frac{3}{n}$$

1st 2nd 3rd .... ith ---- nth

3 x_1 x_2 x_3 x_i 6

(b) x_i is the right endpoint of the ith subinterval

$x_1 = 3+\Delta x$, $x_2 = x_1+\Delta x = 3+2\Delta x$, ..., $x_i = 3+i\cdot\Delta x$.

$$x_i = 3+i\cdot\Delta x = 3+i\cdot\frac{3}{n}, \quad i=1,2,\cdots,n$$

(c) $f(x_i) = 2x_i = 2(3+i\cdot\frac{3}{n})$ is the height of the ith rectangle.

Area of the ith rectangle $f(x_i)\cdot\Delta x = 2(3+i\cdot\frac{3}{n})\cdot\frac{3}{n}$

(d) The sum of all these n rectangles' area

$$\left(f(x_1)\cdot\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x\right) = \sum_{i=1}^{n} f(x_i)\cdot\Delta x = \sum_{i=1}^{n} 2(3+i\cdot\frac{3}{n})\cdot\frac{3}{n} \quad \text{(simplify)}$$

$$= \sum_{i=1}^{n}\frac{18}{n} + \sum_{i=1}^{n}\frac{18}{n^2}\cdot i$$

(Notice that $\frac{18}{n}$, $\frac{18}{n^2}$ are fixed)

$$= \frac{18}{n}\cdot n + \frac{18}{n^2}\cdot\sum_{i=1}^{n} i$$

$$= 18 + \frac{18}{n^2}\cdot\frac{n(n+1)}{2}$$

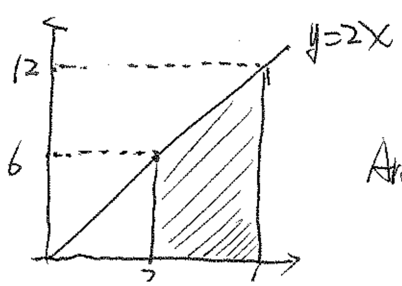
$$= 18 + \frac{9(n+1)}{n}.$$

(e). Take the limit as n goes to $+\infty$

$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\Delta x = \lim_{n\to\infty}\left[18 + \frac{9(n+1)}{n}\right] = 18 + \lim_{n\to\infty}\frac{9n+9}{n}.$$

$$= 18 + \lim_{n\to\infty}\frac{9\cancel{n}}{\cancel{n}} = \boxed{27}$$

Remark: The Area can also be calculated from the graph

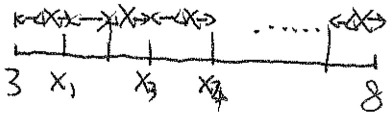


$$\text{Area} = \frac{1}{2}\cdot 3\cdot(6+12) = 27.$$

Hints for webwork

Prob 10. Calculate the area under the curve $f(x)=x^2+6$ from $x=3$ to $x=8$ using Riemann Sum.

(a) $\Delta x = \frac{8-3}{n} = \frac{5}{n}$ length of each subinterval



(b). $x_i = 3 + i\cdot\Delta x = 3+i\cdot\frac{5}{n}$. right endpoint of the ith interval

$x_i = 3+i\cdot\frac{5}{n}$.

(c) Area of the ith rectangle

$$f(x_i)\cdot\Delta x = (x_i^2+6)\cdot\Delta x = \left(\left(3+i\cdot\frac{5}{n}\right)^2+6\right)\cdot\frac{5}{n} = \left(9+\frac{30}{n}\cdot i+\frac{25}{n^2}\cdot i^2+6\right)\cdot\frac{5}{n}.$$

$$= \frac{45}{n}+\frac{150}{n^2}\cdot i+\frac{125}{n^3}\cdot i^2+\frac{30}{n}$$

$$= \frac{75}{n}+\frac{150}{n^2}\cdot i+\frac{125}{n^3}\cdot i^2$$

(d) Riemann Sum

$$\sum_{i=1}^{n} f(x_i)\cdot\Delta x = \sum_{i=1}^{n}\left(\frac{75}{n}+\frac{150}{n^2}\cdot i+\frac{125}{n^3}\cdot i^2\right)$$

$$= \sum_{i=1}^{n}\frac{75}{n} + \sum_{i=1}^{n}\frac{150}{n^2}\cdot i + \sum_{i=1}^{n}\frac{125}{n^3}\cdot i^2$$

$$= \frac{75}{n}\cdot n + \frac{150}{n^2}\cdot\frac{n(n+1)}{2} + \frac{125}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= 75 + 75\cdot\frac{n+1}{n} + \frac{125}{6}\cdot\frac{(n+1)(2n+1)}{n^2}$$

(e) $\text{Area} = \lim_{n\to\infty}\left(75+75\cdot\frac{n+1}{n}+\frac{125}{6}\cdot\frac{(n+1)(2n+1)}{n^2}\right)$

$$= 75 + \lim_{n\to\infty} 75\cdot\frac{\cancel{n+1}}{n} + \lim_{n\to\infty}\frac{125}{6}\cdot\frac{\cancel{(n+1)(2n+1)}}{n^2}$$ Highest order rule.

$$= 75+75+\frac{125}{6}\cdot 2 = \boxed{150+\frac{125}{3}}$$

Prob 3. If $5-10+15-20+25 = \sum_{i=1}^{5} g(i)$, what is $g(i)$?

$5=5\cdot 1$, $-10=5\cdot(-2)$, $15=5\cdot(+3)$, $-20=5\cdot(-4)$, $25=5\cdot 5$

$=5\cdot 1\cdot(-1)^{1-1}$ $\quad =5\cdot 2\cdot(-1)^{2-1}$ $\quad =5\cdot 3\cdot(-1)^{3-1}$, $\quad =5\cdot 4\cdot(-1)^{4-1}$ $\quad =5\cdot 5\cdot(-1)^{5-1}$

$\Rightarrow g(i) = 5\cdot i\cdot(-1)^{i-1}$

Prob 6. $\sum_{i=1}^{75}(-4i^2) = -4\cdot\sum_{i=1}^{75} i^2 = -4\cdot\frac{75\cdot(75+1)\cdot(2\cdot 75+1)}{6} = -4\cdot\frac{75\cdot 76\cdot 151}{6}$

$\sum_{i=1}^{3}(-4i^2) = -4\sum_{i=1}^{3} i^2 = -4\cdot\frac{3\cdot(3+1)\cdot(2\cdot 3+1)}{6} = -4\cdot\frac{3\cdot 4\cdot 7}{6}$

$\sum_{i=4}^{75}(-4i^2) = \sum_{i=1}^{75}(-4i^2) - \sum_{i=1}^{3}(-4i^2) = -4\cdot\frac{75\cdot 76\cdot 151}{6} - \left(-4\cdot\frac{3\cdot 4\cdot 7}{6}\right)$